

# Thermalization of an impurity cloud in a Bose-Einstein condensate.

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We study the thermalization dynamics of an impurity cloud inside a Bose-Einstein condensate at finite temperature, introducing a suitable Boltzmann equation. Some values of the temperature and of the initial impurity energy are considered. We find that, below the Landau critical velocity, the macroscopic population of the initial impurity state reduces its depletion rate. For sufficiently high velocities the opposite effect occurs. For appropriate parameters the collisions cool the condensate. The maximum cooling per impurity atom is obtained with multiple collisions.

In a Bose-Einstein condensate (BEC) many atoms populate the same quantum state. This system is, therefore, suitable to realize an atom laser [1]. That is obtained creating atoms in an untrapped or antitrapped level by means of a rf or Raman outcoupling. Elastic collisions between impurity atoms and the parent BEC have been observed at MIT [2]. These collisions can deplete [3] and heat the outcoupled atoms. In this work we study the thermalization dynamics of the impurity atoms. We find the following main results: i) we evaluate the impurity thermalization and depletion time for some values of the temperature and of the initial impurity energy. The thermalization is not suppressed below the Landau critical velocity [4] for a finite condensate temperature  $T$ , according to a previous work [5]; ii) below the Landau critical velocity the macroscopic population of the atom laser state slows down the impurity depletion, whereas, for sufficiently high velocities the macroscopic population enhances the scattering rate, as observed in Ref. 2; iii) the collisions cool the condensate if the impurity initial energy  $\epsilon_{in}$  is below its thermal equilibrium energy, or even if this condition is not fulfilled, provided the thermal energy is sufficiently smaller than the chemical potential. These effects can be observed creating a succession of atom laser packets and observing a reduction of the impurity depletion rate in the last packets. The succession of packets can be used to reduce the condensate temperature, as proposed in Ref. 5. In Ref. 5 we evaluated the scattering rate of an impurity inside a condensate at finite temperature and found that for suitable initial velocities the impurity acquired energy, on average, after the first collision. However processes with multiple collisions were not considered. Here we show that the maximum cooling per impurity atom is obtained, for  $T$  not too small, at thermal equilibrium, that is, with multiple collisions.

To simplify the problem we consider an homogeneous condensate, in fact it is meaningful in a trapped system in the sense of the Thomas-Fermi approximation. If the number of outcoupled atoms is sufficiently low, the initial thermal energy of the impurities can be neglected. The first cooling is obtained by the Raman scattering, that creates a cloud with a smaller temperature [6]. We first introduce a Boltzmann equation for an impurity atom. A similar equation in momentum space is obtained, for a single particle and  $T = 0$ , in Ref. 7. It is easy to show that the system goes toward the Boltzmann equi-

librium distribution. We extend our discussion to a cloud of impurity atoms introducing suitable bosonic stimulation terms. We obtain an equation of the same type of the quantum Boltzmann equation [8, 9, 10]. It is verified that the Bose-Einstein distribution is the steady state solution.

Let us consider an impurity atom in an uniform condensate at temperature  $T$ . If the impurity energy is  $E_i$ , the probability that the particle scatters in a state with energy  $E_f$  is  $\Gamma(E_f, E_i) = \Gamma_1(E_f, E_i) + \Gamma_2(E_f, E_i)$ , where [5]

$$\Gamma_{1,2}(E_f, E_i) = n_0 \left( \frac{2\hbar a}{M} \right)^2 \int dq d\Omega q^2 S(q) \times \quad (1)$$

$$\delta \left( \frac{E_i - E_f}{\hbar} \mp \omega_q^B \right) \delta \left( E_f - \frac{\hbar^2 (\vec{k}_i \mp \vec{q})^2}{2M} \right) \frac{\pm 1}{1 - e^{\mp \beta \hbar \omega_q^B}},$$

$\hbar \vec{k}_i$ ,  $1/\beta \equiv k_B T$ ,  $S(q) \equiv \omega_q^0 / \omega_q^B$ ,  $\hbar \omega_q^0$  and  $\hbar \omega_q^B \equiv \sqrt{\hbar \omega_q^0 (\hbar \omega_q^0 + 2\mu)}$  being, respectively, the initial impurity momentum, the condensate temperature in energy unit, the static structure factor and the energies of a free particle and of a Bogoliubov quasiparticle of momentum  $q$ .  $M$ ,  $\mu$  and  $a$  are the atomic mass, the chemical potential and the scattering length for s-wave collisions between the impurity atoms and the condensate ones.  $\mu$  is equal to  $n_0 g$ , where  $n_0$  is the condensate density and  $g = 4\pi \hbar^2 a_0 / M$ ,  $a_0$  being the condensate s-wave scattering length.

$\Gamma_1$  and  $\Gamma_2$  are associated with dissipative and cooling processes, respectively. In the first case condensate phonons are created, in the second one they are annihilated [5]. With a little of algebra we find that  $\Gamma(E_f, E_i) = 2(\gamma/\mu)G(2E_f/\mu, 2E_i/\mu)$ , where  $\gamma = 4\pi n_0 a^2 c$ ,  $c \equiv \sqrt{\mu/M}$  being the sound velocity, and

$$G(\epsilon_f, \epsilon_i) = \frac{1}{\sqrt{\epsilon_i}} \left[ 1 - \frac{1}{\sqrt{1 + \Delta \epsilon^2/4}} \right] \times \frac{W(\epsilon_f, \epsilon_i)}{1 - e^{\bar{\beta}(\epsilon_f - \epsilon_i)}},$$

with  $\bar{\beta} = \beta\mu/2$  and  $\Delta\epsilon = \epsilon_f - \epsilon_i$ .  $W(\epsilon_f, \epsilon_i)$  is 1 when  $\epsilon_i \geq \epsilon_f \geq 1/\epsilon_i$ ,  $-1$  when  $\epsilon_f > \epsilon_i \geq 1/\epsilon_f$  and zero elsewhere. The Boltzmann equation is

$$\frac{dP(\epsilon)}{d\tau} = \int G(\epsilon, \bar{\epsilon})P(\bar{\epsilon})d\bar{\epsilon} - \int G(\bar{\epsilon}, \epsilon)P(\epsilon)d\bar{\epsilon}, \quad (2)$$

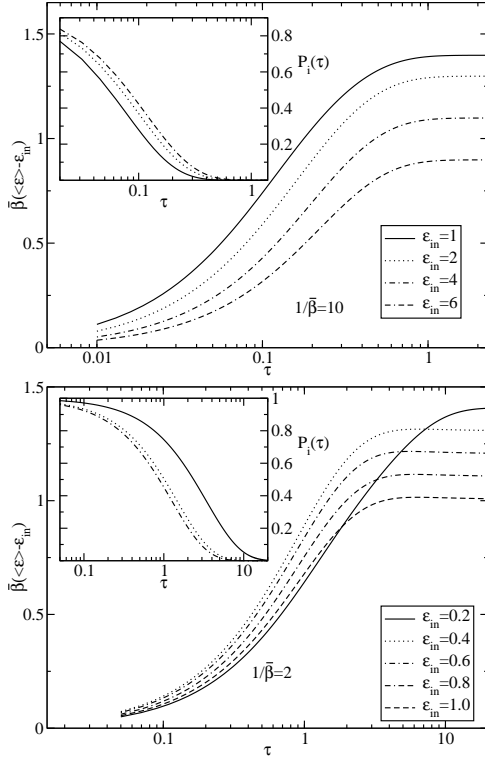


FIG. 1: Average energy acquired by the impurity in  $k_B T$  unit as a function of  $\tau$ , for some values of  $\epsilon_{in}$ . In the inset is reported the relative density  $P_i(\tau)$  as a function of  $\tau$ . (a)  $1/\bar{\beta} = 10$ , i.e.  $k_B T = 5\mu$ . (b)  $1/\bar{\beta} = 2$ , i.e.  $k_B T = \mu$ .

where  $P(2E/\mu)$  is the density probability per unit energy and  $\tau \equiv \gamma t$ . It is easy to demonstrate that  $P(\epsilon) \propto \sqrt{\epsilon} e^{-\bar{\beta}\epsilon}$  is a steady solution of Eq. (2).

The system dynamics depends exclusively on the product  $\beta\mu = \mu/(k_B T)$  and the initial probability distribution. The condensate density and the scattering length are present in the rate factor  $\gamma$  and, therefore, they influence only the time scale of the dynamics.

In Fig. 1a we report the average energy acquired by the impurity and lost by the condensate thermal atoms, in  $k_B T$  unit, i.e.  $\bar{\beta}[\langle\epsilon\rangle - \epsilon_{in}] = \beta[\langle E\rangle - E_{in}]$ . We have considered  $1/\bar{\beta} = 10$  and some values ( $\geq 1$ ) of the initial impurity energy. We can note that  $\langle E\rangle = (3/2)k_B T$  for  $\tau \rightarrow \infty$ , that is the average energy of a free particle at the equilibrium. In contrast, if we neglect the collective modes, a thermal particle of the condensate has an average energy equal to  $\sim 0.514(3/2)k_B T$ , that is, about one half of the classical value. It is easy to demonstrate that the number of lost thermal atoms  $\delta N$ , when the condensate energy decreases by  $\delta E$ , is  $\sim 0.78\delta E/(k_B T)$ . Therefore, if the impurity acquires an energy of  $\sim 1.3k_B T$  a thermal atom jumps into the condensate state, balancing the loss due to the impurity creation. A nearly complete thermalization occurs for  $\tau \simeq 0.5$ , that is, for  $t_{th} \simeq 1/(2\sigma_0 n_0 c)$ , where  $\sigma_0 \equiv 4\pi a^2$ . For a  $^{23}\text{Na}$  condensate and a typical density of  $10^{14} \div 10^{15} \text{cm}^{-3}$  ( $T \sim 350 \div 3500 \text{ nK}$  for  $1/\bar{\beta} = 10$ ) we have  $t_{th} \simeq 10 \div 0.3 \text{ ms}$ .

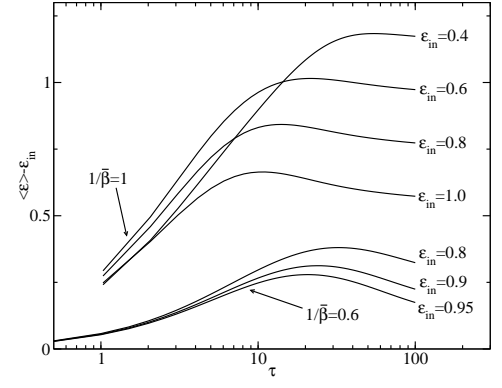


FIG. 2: Average energy acquired by the impurity in  $\mu/2$  unit as a function of  $\tau$ , for some values of  $\epsilon_{in}$  and for  $1/\bar{\beta} = 0.6 \div 1$ .

We can see that the thermalization time is independent on the initial velocity. In fact, the impurity loses quickly the memory of its initial energy. In the inset of Fig. 1a we report the relative density  $P_i(\tau) \equiv P(\epsilon_{in}, \tau)/P(\epsilon_{in}, 0)$  as a function of  $\tau$ . The initial state population decreases by  $60 \div 70$  per cent for  $\tau \sim 0.1$ , i.e. for  $t = 0.06 \div 2 \text{ ms}$ . We note that the depletion rate is smaller for a higher initial velocity, contrary to the zero temperature case. In fact, for  $T = 0$  we expect that the scattering rate grows upon increasing the initial velocity. At finite temperatures, to reduce the laser mode depletion, it is convenient to have a large initial velocity, because the scattering rate is smaller and, furthermore, the impurity leaves the condensate earlier. For a sufficiently high velocity we expect that the scattered impurity fraction is not enhanced by the finite temperature.

If the condensate temperature is reduced, the scattering rate decreases and approaches the zero temperature value. For  $T = 0$  only the spontaneous scattering contributes and when  $\epsilon_{in} < 1$  the impurity thermalization is absent because of the condensate superfluidity. For  $k_B T = \mu$  ( $\bar{\beta} = 2$ ) the temperature can be still important. In fact, we can see in Fig. 1b that a nearly complete thermalization occurs for  $\tau \sim 2$ , that, for  $n_0 = 10^{15} \text{cm}^{-3}$ , corresponds to  $t \sim 1.2 \text{ ms}$ . The initial velocity is below the Landau value, so the first impurity scattering is due to the finite temperature. The function  $G(\epsilon_f, \epsilon_i)$  is equal to zero for  $\epsilon_f < 1/\epsilon_i$ , therefore, if the initial energy is too low, the particle jumps in a state with a large energy. However the probability rate of this process is small, because of the presence of the exponential in  $G$ . So we have a slowing down of the thermalization rate (see Fig. 1b, solid line) when the initial velocity is very low. For  $\epsilon_{in} \geq 0.4$  the cut-off of the constraint  $\epsilon_f > 1/\epsilon_i$  has a minor effect and the thermalization time becomes again independent on the initial velocity. Also the depletion rate of the relative density  $P_i(\tau)$  is reduced by the cut-off for  $\epsilon_{in} = 0.2$ . Increasing  $\epsilon_{in}$  the depletion rate grows, however for  $\epsilon_{in} > 0.6$  it slows down again, as it occurs for  $\bar{\beta} = 10$  (see Fig. 1a). For  $\epsilon_{in} = 0.4$  and  $n_0 = 10^{14} \div 10^{15} \text{cm}^{-3}$  the initial impurity velocity

is  $\sim 0.3 \div 1 \text{ cm/s}$ , therefore, the average distance before the first collision ( $\tau \sim 1$ ) is about  $4 \cdot (10^{-2} \div 10^{-3}) \text{ mm}$ . The radius of a condensate with the same central density is  $\sim 0.5 \div 2 \cdot 10^{-1} \text{ mm}$  when the trap frequency is  $20 \text{ Hz}$ . To contrast the gravitational acceleration of the impurities in the radial direction of a horizontal cylindrical trap we can use a gaussian red-detuned laser beam. We can set this beam and the magnetic field in a such way to permit that the impurities leave the condensate after the thermalization time.

In Fig. 2 we report the same curves of Fig. 1, but the energy is in  $\mu/2$  unit and  $1/\bar{\beta} = 0.6 \div 1$ . In this case the energy reaches a maximum at an intermediate time. For  $1/\bar{\beta} = 1$  and  $\epsilon_{in} = 0.6 \div 1.0$  the maximum is reached at  $\tau = 10 \div 20$ , that, for  $n_0 = 10^{15} \text{ cm}^{-3}$ , corresponds to  $t = 6 \div 10 \text{ ms}$ . We find that the evolution of the initial state population is nearly independent on  $\epsilon_{in}$ . At  $\tau = 10$  the initial state is depleted by 40%. To cool the condensate it is not necessary to reach the maximum, but for smaller times we obtain a minor reduction of the condensate temperature per atom. Note that for  $1/\bar{\beta} = 0.6$  the equilibrium thermal energy is 0.9, in  $\mu/2$  unit. Therefore, for  $\epsilon_{in} = 0.9 \div 0.95$  the initial energy is equal or higher than the equilibrium value. Even so, at an intermediate time the impurity acquires energy and the condensate cools. That occurs because when the velocity is below the critical value the first scattering can only enhance the impurity energy.

Let us suppose now that there are many impurities, then many atoms can be in the same energy level and, therefore, it is necessary to consider the stimulation effects. If we suppose that the initial impurity velocity is isotropic, we can make the following modification in Eq. (2)

$$\frac{dp(\epsilon)}{d\tau} = [p(\epsilon)/\rho(\epsilon) + 1] \int G(\epsilon, \bar{\epsilon}) p(\bar{\epsilon}) d\bar{\epsilon} - \int [p(\bar{\epsilon})/\rho(\bar{\epsilon}) + 1] G(\bar{\epsilon}, \epsilon) p(\epsilon) d\bar{\epsilon} \equiv B(p), \quad (3)$$

where  $p(\epsilon) = P(\epsilon)/V$ ,  $V$  being the system volume, and  $\rho(2E/\mu) = (2m)^{3/2} \hbar^{-3} (2\pi)^{-2} \sqrt{E}$  is the level density per unit volume [11, 12]. It is easy to demonstrate that the Bose-Einstein distribution  $p(\epsilon) \propto \sqrt{\epsilon} (e^{\beta(\epsilon - \bar{\mu})} - 1)^{-1}$  is a steady solution of Eq. (3),  $\bar{\mu}$  being the normalized chemical potential of the impurity cloud.

We isolate in  $p(\epsilon)$  the contribution of the initial populated state with energy  $\epsilon_{in}$ . The associated distribution, that is very narrow, has a width equal to  $\delta\epsilon \equiv 2/(V\rho\mu)$ . We indicate with  $p_i$  its height.

Eq. (3) becomes

$$\frac{dp(\epsilon)}{d\tau} = B(p) + \delta\epsilon [p(\epsilon)/\rho(\epsilon) + 1] G(\epsilon, \epsilon_{in}) p_i - \delta\epsilon [p_i/\rho(\epsilon_{in}) + 1] G(\epsilon_{in}, \epsilon) p(\epsilon) \quad (4)$$

$$\frac{dp_i}{d\tau} = [p_i/\rho(\epsilon_{in}) + 1] \int G(\epsilon_{in}, \bar{\epsilon}) p(\bar{\epsilon}) d\bar{\epsilon} - \int [p(\bar{\epsilon})/\rho(\bar{\epsilon}) + 1] G(\bar{\epsilon}, \epsilon_{in}) p_i d\bar{\epsilon}. \quad (5)$$

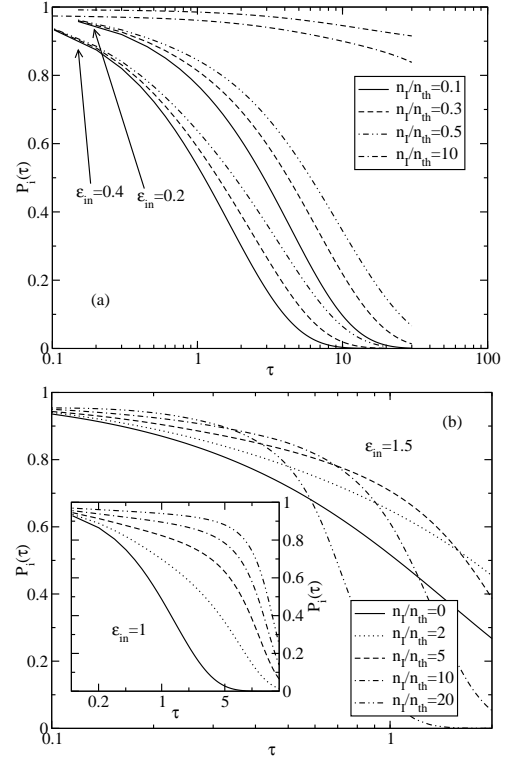


FIG. 3: Relative density  $P_i(\tau)$  as a function of  $\tau$  for  $1/\bar{\beta} = 2$  and some values of  $n_i/n_{th}$ , with (a)  $\epsilon_{in} = 0.2 \div 0.4$ , (b)  $\epsilon_{in} = 1.5$  and  $\epsilon_{in} = 1$  in the inset.

The quantities  $\delta\epsilon G(\epsilon_{in}, \epsilon) p(\epsilon)$  and  $\int G(\epsilon_{in}, \bar{\epsilon}) p(\bar{\epsilon}) d\bar{\epsilon}$  in the first and second equation, respectively, can be neglected. Therefore Eqs. (4,5) become

$$\frac{dp(\epsilon)}{d\tau} = B(p) + \frac{2}{\mu} \{ [p(\epsilon)/\rho(\epsilon) + 1] G(\epsilon, \epsilon_{in}) - G(\epsilon_{in}, \epsilon) p(\epsilon)/\rho(\epsilon_{in}) \} n_i \quad (6)$$

$$\frac{dn_i}{d\tau} = -\gamma_i(p) n_i, \quad (7)$$

where  $n_i$  is the spatial density of the impurity cloud and

$$\gamma_i(p) = \int [p(\bar{\epsilon})/\rho(\bar{\epsilon}) + 1] G(\bar{\epsilon}, \epsilon_{in}) d\bar{\epsilon} - 1/\rho(\epsilon_{in}) \int G(\epsilon_{in}, \bar{\epsilon}) p(\bar{\epsilon}) d\bar{\epsilon} = \int \left\{ G(\bar{\epsilon}, \epsilon_{in}) - \frac{W(\epsilon_{in}, \bar{\epsilon})}{\rho(\epsilon_{in})} \left[ 1 - \frac{1}{\sqrt{1 + \Delta\epsilon^2/4}} \right] p(\bar{\epsilon}) \right\} d\bar{\epsilon}, \quad (8)$$

$\Delta\epsilon$  being  $\epsilon_{in} - \bar{\epsilon}$ . The first term of  $\gamma_i$  is the depletion rate of the initial state when the stimulated effects of the impurities are negligible. The second term, that does not depend directly on the temperature, is proportional to the density  $p(\bar{\epsilon})$  and it is important when there are many impurities. This term is negative when  $\epsilon_{in} < 1$ , therefore in this case it reduces the depletion rate  $\gamma_i$ . In fact, the scattered atoms can have a high probability to scatter back into the original macroscopically populated state.

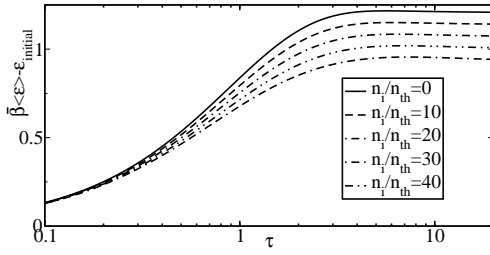


FIG. 4: Same as with Fig. 1, but for an impurity cloud. Some values of  $n_I$  are considered, for  $1/\bar{\beta} = 2$  and  $\epsilon_{in} = 0.6$ .

On the contrary, when  $\epsilon_{in} > 1$  the second integrand is negative for  $\bar{\epsilon} > \epsilon_{in}$  and positive for  $\bar{\epsilon} < \epsilon_{in}$ ; if  $\epsilon_{in}$  is sufficiently high the impurity stimulation effect enhances the depletion rate, as it has been experimentally observed [2]. In our calculations we have considered the condensate as a thermal reservoir, however if we have many impurities the collisions can cool or warm the condensate, consequently the depletion rate can decrease or increase. Our approximation is reasonable when the density of outcoupled atoms is small with respect to the density of thermal atoms  $n_{th}$  or, otherwise, for sufficiently small times, when the collided fraction is small. We have approximatively, if we neglect the condensate collective modes, [11]

$$n_{th} = (4\pi^2 \hbar^3)^{-1} m^{3/2} (\mu/\bar{\beta})^{3/2} \quad (9)$$

In Fig. 3a we report the relative density  $P_i(\tau)$  for  $1/\bar{\beta} = 2$ ,  $\epsilon_{in} = 0.2 \div 0.4$  and some values of  $n_I/n_{th} \equiv n_i(0)/n_{th}$ . It is easy to demonstrate that these curves are independent on  $\mu$  and, therefore, on  $a$  and  $n_0$ . In fact, with  $p(\epsilon) \propto \sqrt{\mu}$  and  $n_I \propto \mu^{3/2}$ , the Boltzmann equations are invariant with respect to modifications of the chemical potential. For  $n_0 = 10^{14} \text{cm}^{-3}$  and  $n_0 = 10^{15} \text{cm}^{-3}$  we have  $n_{th} \simeq 1.1 \cdot 10^{12} \text{cm}^{-3}$  and  $n_{th} \simeq 3.5 \cdot 10^{13} \text{cm}^{-3}$ , respectively. We can see that for  $n_I/n_{th} = 10$  the depletion rate is considerably reduced by the stimulation effects. In

Fig. 3b we plot the same curves but for  $\epsilon_{in} = 1$  (inset) and 1.5, respectively. These curves are reliable only at the initial times because of the cooling of the condensate. Furthermore, if collisions create many thermal impurity atoms, a lot of impurities could condensate into the state with zero energy, but our equations are not suitable to describe this phenomenon, because this state has to be treated separately from the energy integral of Eq. (6). If the number of scattered atoms is lower than  $n_{th}$  we can suppose that this condensation does not occur. Note that the depletion rate is lowered also for  $\epsilon_{in} = 1.5$ , that is above the Landau value. Instead, for  $\epsilon_{in} = 2$  we find that the depletion rate is enhanced. Increasing the temperature the threshold of  $\epsilon_{in}$  increases.

Finally, we report in Fig. 4 the same curves of Fig. 1, but accounting for the stimulation effects of the impurity cloud. We have considered some values of  $n_I$ , for  $1/\bar{\beta} = 2$  and  $\epsilon_{in} = 0.6$ . The maximum acquired energy is reduced increasing  $n_I$ .

In conclusion, we have evaluated the thermalization time of impurity atoms inside a condensate at finite temperature and we have found that their thermalization, that occurs also below the Landau critical velocity, can cool the condensate. In fact, a *small* Raman or rf outcoupling creates a condensate in another level with a lower temperature [6], that cools sympathetically the parent condensate. This effect can be studied experimentally creating untrapped impurities whose gravitational acceleration is compensated by a detuned laser field. The cooling effect could be used to remove the residual thermal atoms. Furthermore, we have accounted for the stimulation effects of the impurity cloud and we find that below the Landau critical velocity the macroscopic population of the initial state can reduce drastically its depletion rate. For sufficiently high velocities the opposite effect occurs, as observed in Ref. 2. The threshold velocity depends on the condensate temperature. Our analysis is also important for the development of intense atom lasers.

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